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The decomposability of the matrix with two determinantal regional components ☆

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ABSTRACT

The ray of a complex number a is either 0 or $a/|a|$ depending on whether a is 0 or nonzero. The ray pattern of a complex matrix A, denoted by $\text{ray}(A)$, is the matrix obtained by replacing each entry of A with its ray. The determinantal region of a square matrix A, denoted by R_A, is the set of the determinants of all the complex matrices with the same ray pattern as A. A connected component of the set $R_A \setminus \{0\}$ is called a determinantal regional component of A. The number of determinantal regional components of R_A is denoted by $n_R(A)$. It was proved in Shao et al. [Jia-Yu Shao, Yue Liu, Ling-Zhi Ren, The inverse problems of the determinantal regions of ray pattern and complex sign pattern matrices, Linear Algebra Appl. 416 (2006) 835–843] that $n_R(A) \leq 2$ for any complex square matrix A. When $n_R(A) = 2$, the two determinantal regional components are either two opposite open rays or two opposite open sectors with the angle no more than π. In this paper, we prove that any square matrix A with $n_R(A) = 2$ is partly decomposable if one of its determinantal regional components is an open sector with the angle less than π. As a main graph theoretical technique, we also discuss a property of strongly connected digraphs.

1. Introduction

The concept of ray pattern matrices is a natural generalization of that of the sign pattern matrices from the real case to the complex case [1,2].

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Let a be a complex number. Its ray, denoted by $\mathrm{ray}(a)$, is defined as

$$\mathrm{ray}(a) = \begin{cases} a/|a|, & \text{if } a \neq 0; \\ 0, & \text{if } a = 0. \end{cases}$$

Let A be a complex matrix. The matrix obtained from A by replacing its entries with their rays is called the ray pattern of A, denoted by $\mathrm{ray}(A)$. The set of all matrices which have the same ray pattern as A is called the ray pattern class of A, denoted by $Q_R(A)$. Namely

$$Q_R(A) = \{B|\mathrm{ray}(B) = \mathrm{ray}(A)\}.$$

The matrix A is said to be ray nonsingular if A is square and all the matrices in $Q_R(A)$ are nonsingular.

The determinantal region of a complex square matrix A is defined to be

$$R_A = \{\det B|B \in Q_R(A)\}.$$

It is easy to see that A is ray nonsingular if and only if $0 \notin R_A$.

Some fundamental work on the determinantal regions has been done in [1–3]. It was shown in [1] that for an arbitrary square matrix A, R_A is a connected set in the complex plane and closed under multiplication by a positive scalar. In [3], it is proved that $R_A \backslash \{0\}$ is either an open set in the complex plane $\mathbb{C}$ or contained in a line through the origin. Thus $R_A \backslash \{0\}$ is either a union of open sectors or a union of at most two open rays. An open sector from α to β (where α and β are two real numbers with $\alpha < \beta$), denoted by $S_{(\alpha,\beta)}$, is defined to be:

$$S_{(\alpha,\beta)} = \left\{re^{i\theta} \mid r > 0,\ \alpha < \theta < \beta\right\},$$

and $\beta - \alpha$ is called the angle of this open sector. The angle of an open sector F is also denoted by $\mathrm{ang}(F)$. Similarly we can define $S_{[\alpha,\beta]}$.

Let $b_U(R_A)$ be the intersection of the boundary of R_A and the unit circle U, and $T(A)$ be the set of distinct nonzero terms in the determinantal expansion of the matrix $\mathrm{ray}(A)$. $T(A)$ is also called the *transversal set* of A. The following theorem concerning the boundary of R_A is given in [5], and will be used in later proofs.

Theorem 1.1 [5]. *Let A be a complex square matrix, then $b_U(R_A) \subseteq T(A)$.*

The number of connected components of $R_A \backslash \{0\}$ is denoted by $n_R(A)$. In this paper we call a connected component of $R_A \backslash \{0\}$ a ***determinantal regional component***. It has been proved in [4] that $n_R(A) \leqslant 2$ for any complex square matrix A. If $n_R(A) = 2$, then $R_A = F \cup \{0\} \cup (-F)$, where F is either an open ray or an open sector with the angle no more than π. In Section 3, we will prove that the matrix A is partly-decomposable if $R_A = F \cup \{0\} \cup (-F)$ where F is an open sector with $\mathrm{ang}(F) < \pi$.

2. Preliminaries

In this section we give some basic definitions and propositions which will be used in the proof of our main result. We will define a kind of cycle sequence (called "cycle chain") in a digraph and show that cycle chains exist in every strongly connected digraph. We also obtain a recurrence formula for calculating the determinant of the adjacency matrix of an arc-weighted digraph induced by the arc set of a cycle chain (together with all the loops).

The matrix A is called ***partly-decomposable*** if there exist permutation matrices P and Q such that

$$PAQ = \begin{pmatrix} X & 0 \\ Z & Y \end{pmatrix},$$

where X and Y are non-empty square matrices. A is called ***fully-indecomposable*** if it is not partly-decomposable.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two sub-digraphs of a digraph G, in this paper we write $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

Definition 2.1. An ordered triple $W = (V, E, w)$ is called an ***arc-weighted digraph***, where $V = \{v_1, \ldots, v_n\}$ is the vertex set of W, E is the arc set of W which is a subset of $V \times V$; w is a mapping from the arc set E to the set $\mathbb{C}\backslash\{0\}$ of nonzero complex numbers, which is called the weight function of W. The directed graph (V, E) is called the ***underlying digraph*** of W, and is denoted by $D(W)$.

The weight of a path or a cycle P in an arc-weighted digraph, denoted by $w(P)$, is the product of the weights of all the arcs of P. The ray of P, denoted by $\mathrm{ray}(P)$, is the ray of the complex number $w(P)$. Thus we have $w(P) = |w(P)|\mathrm{ray}(P)$.

Definition 2.2. Let $W = (V, E, w)$ be an arc-weighted digraph with $V = \{v_1, \ldots, v_n\}$. Define the matrix $A(W) = (a_{jk})_{n\times n}$ as:

$$a_{jk} = \begin{cases} w\left((v_j, v_k)\right), & (v_j, v_k) \in E; \\ 0, & (v_j, v_k) \notin E. \end{cases}$$

Then $A(W)$ is called the ***adjacency matrix*** of W.

From the definition we know that the adjacency matrix is not uniquely determined by the digraph W itself. It also depends on the ordering of the vertices. But any two adjacency matrices of the same digraph are permutation similar to each other, so they have the same determinant and the same determinantal region.

Definition 2.3. Let W be an arc-weighted digraph, and $A(W)$ be an adjacency matrix of W. Then the determinant of $A(W)$ is also called the determinant of W, denoted by $\det W$.

The following definition extends the idea of ray pattern class from square matrices to arc-weighted digraphs.

Definition 2.4. Let $W = (V, E, w)$ be an arc-weighted digraph. Let

$$Q_R(W) = \{W' | \mathrm{ray}(A(W')) = \mathrm{ray}(A(W))\}.$$

Furthermore, we define

$$R_W = \{\det W' | W' \in Q_R(W)\},$$

then R_W is called the ***determinantal region*** of the arc-weighted digraph W.

Definition 2.5. Let $A = (a_{jk})_{n\times n}$ be a square matrix of order n, $D(A) = (V_A, E_A)$ be the associated digraph of A. Define a weight function w_A as: $w_A(e) = a_{jk}$ for each arc $e = (j, k) \in E_A$. Then the resulting arc-weighted digraph $W(A) = (V_A, E_A, w_A)$ is called the arc-weighted digraph of A.

We use $\langle n\rangle$ to denote the set $\{1, \ldots, n\}$ in this paper.

A digraph D is called a *permutation digraph* if the in-degree and out-degree of each vertex of D are both 1. It is easy to see that a digraph is a permutation digraph of order n if and only if it is a vertex disjoint union of cycles and loops with total length n.

It is well-known that each nonzero term in the determinantal expansion of a square matrix A can be expressed in such a way:

$$(\mathrm{sgn}\,\sigma)a_{1\sigma(1)}\ldots a_{n\sigma(n)} = (-1)^n \prod_{C\in L_\sigma} (-w(C)), \tag{2.1}$$

where $\sigma \in S_n$ is a permutation on the set $\langle n\rangle$, L_σ is the set of the cycles and loops contained in the spanning permutation sub-digraph of $D(A)$ induced by the n arcs $\{(i, \sigma(i)) | i = 1, \ldots, n\}$, and $w(C)$ is the weight of the cycle C.

Suppose A is a square matrix of order n. A set of disjoint cycles and loops in $W(A)$ with the total length n is called a *permutation cycle set* (since the cycles and loops just form a spanning permutation sub-digraph) of $W(A)$. Let $\mathcal{C}^*(A)$ be the class of all the permutation cycle sets of $W(A)$. Then together with (2.1), we have the following lemma.

Lemma 2.1. *Let A be a square matrix of order n, then*

$$\det A = (-1)^n \sum_{L \in \mathcal{C}^*(A)} \prod_{C \in L} (-w(C)). \tag{2.2}$$

Next we introduce a graph theoretical concept, called "cycle chain", which is a main technique used in this paper. We begin with the following definition.

Definition 2.6. A sequence of pairwisely distinct cycles $C_1, \ldots, C_n$ in a digraph is called a ***cycle chain*** if it satisfies:

(1) Two cycles C_j and C_k have common vertices if and only if $|j-k| \leqslant 1$;
(2) There is no other cycle in the digraph induced by the arc set $\bigcup_{i=1}^{n} E(C_i)$.

Then n is called the length of the cycle chain, and $C_1,\ C_n$ are called the two end cycles.

It is easy to see that the digraph induced by the arc set of a cycle chain is strongly connected, and when the length of the cycle chain is 1, the digraph is just a single cycle.

The next lemma shows the existence of cycle chains in strongly connected graphs.

Lemma 2.2. *Let D be a strongly connected digraph, C_1 and C_2 be two vertex disjoint cycles in D. Then there exists a cycle chain with C_1 and C_2 as its two end cycles.*

Proof. Since D is strongly connected, there is a shortest path P_1 from C_1 to C_2, thus it is internally vertex disjoint with C_1 and C_2. Similarly we can find a shortest path P_2 from C_2 to C_1 which is also internally vertex disjoint with C_1 and C_2. Denote the sub-digraph $C_1 \cup C_2 \cup P_1 \cup P_2$ by D'. It is easy to see that D' is also strongly connected. Next we will show that there exists a cycle chain in D' joining C_1 and C_2 by using induction on the number of arcs.

Denote the initial and the terminal vertex of P_1 by a_1 and a_1', the initial and the terminal vertex of P_2 by a_2' and a_2 (a_1 and a_2 may be the same, a_1' and a_2' may be the same, too).

If P_1 and P_2 are internally vertex disjoint, then we take the cycle E_1 to be:

$$a_1 \xrightarrow{P_1} a_1' \xrightarrow{C_2} a_2' \xrightarrow{P_2} a_2 \xrightarrow{C_1} a_1.$$

It is easy to verify that C_1, E_1, C_2 is a cycle chain in D'.

Now we only need to consider the case when P_1 and P_2 have some internal vertices in common. Denote the first common internal vertex on P_1 by b_1, and the last common internal vertex on P_2 by b_2 (b_1 and b_2 may be the same).

Denote the directed cycle $a_1 \xrightarrow{P_1} b_1 \xrightarrow{P_1} b_2 \xrightarrow{P_2} a_2 \xrightarrow{C_1} a_1$ by E_1, then E_1 and C_2 are disjoint. In D', choose a shortest subpath P_1' of P_1 from E_1 to C_2, then P_1' is internally vertex disjoint with E_1 and C_2. Similarly, we can find a shortest subpath P_2' of P_2 from C_2 to E_1 and internally vertex disjoint with E_1 and C_2. Denote the digraph $E_1 \cup C_2 \cup P_1' \cup P_2'$ by D''. Then D'' is strongly connected, too.

In the strongly connected digraph D'', E_1 and C_2 are two disjoint cycles. By induction on the number of the arcs of D'', we know that there exists a cycle chain with E_1 and C_2 as its two end cycles. Denote the cycle chain by $E_1, E_2, \ldots, E_q, C_2$.

Since C_1 is internally vertex disjoint with P_1 and P_2, it is not difficult to verify that C_1 and $E_2, \ldots, E_q$ are vertex disjoint, then $C_1, E_1, \ldots, E_q, C_2$ is just the cycle chain we want. $\square$

The following lemma will be used to calculate the determinant of an arc-weighted cycle chain digraph (together with all the loops).

Lemma 2.3. *Let A be a square matrix with order n and all the diagonal elements -1. Suppose that $C_1, \ldots, C_q$ is a cycle chain in the arc-weighted digraph $W(A)$. Denote by N_0 the arc-weighted digraph consisting of n loops with weights -1. For $j \in \langle q \rangle$, let $N_j = N_{j-1} \cup C_j$. Then we have*

$$\det N_j = -w(C_j)\cdot \det N_{j-2} + \det N_{j-1} \quad (j = 2, \ldots, q),$$

where $\det N_0 = (-1)^n$, $\det N_1 = (-1)^n \cdot (1 - w(C_1))$.

Proof. $\det N_0 = (-1)^n$ is obvious.

For N_1, we know that there are only two permutation cycle sets in $\mathcal{C}^*(A(N_1))$, denoted by L_1 and L_2, where L_1 is the permutation cycle set consisting of all the loops, while L_2 is the permutation cycle set consisting of C_1 and all the loops outside C_1. By Lemma 2.1 we have:

$$\det N_1 = (-1)^n \cdot \prod_{C \in L_1} (-w(C)) + (-1)^n \cdot \prod_{C \in L_2} (-w(C)) = (-1)^n \cdot (1 - w(C_1)).$$

For $2 \leqslant j \leqslant q$, the permutation cycle sets in $\mathcal{C}^*(A(N_j))$ can be divided into two classes: the permutation cycle sets in the first class are the ones not containing the cycle C_j, it is easy to verify that this class is just equal to $\mathcal{C}^*(A(N_{j-1}))$; the other class consists of the permutation cycle sets containing the cycle C_j, denoted by $\mathcal{C}_2^*(A(N_j))$. For the second class, if we replace C_j in each permutation cycle sets by the loops on C_j, the resulting class is just $\mathcal{C}^*(A(N_{j-2}))$. Denote all the loops on C_j by R_j. Since the weight of all loops are -1, by Lemma 2.1, we have

$$\begin{aligned}
\det N_j &= (-1)^n \left(\sum_{L \in \mathcal{C}^*(A(N_{j-1}))} \prod_{C \in L} (-w(C)) + \sum_{L \in \mathcal{C}_2^*(A(N_j))} \prod_{C \in L} (-w(C)) \right) \\
&= \det N_{j-1} - w(C_j) \cdot (-1)^n \cdot \sum_{L \in \mathcal{C}_2^*(A(N_j))} \prod_{C \in L \setminus \{C_j\}} (-w(C)) \\
&= \det N_{j-1} - w(C_j) \cdot (-1)^n \sum_{(L \setminus \{C_j\}) \cup R_j \in \mathcal{C}^*(A(N_{j-2}))} \prod_{C \in (L \setminus \{C_j\}) \cup R_j} (-w(C)) \\
&= -w(C_j) \cdot \det N_{j-2} + \det N_{j-1}.
\end{aligned}$$

The proof is completed. $\square$

3. The main result

In this section, we will prove that A is partly decomposable if $R_A = F \cup \{0\} \cup (-F)$, where F is an open sector with $\operatorname{ang}(F) < \pi$.

Let $A = (a_{jk})_{m \times n}$ and $B = (b_{jk})_{m \times n}$ be two matrices. If for any $b_{jk} \neq 0$ we have $\operatorname{ray}(b_{jk}) = \operatorname{ray}(a_{jk})$, then B is called ***ray majorized*** by A, and denoted by $B \lesssim A$.

The next lemma is a starting point of our discussion, and will be used repeatedly.

Lemma 3.1. *Let A and B be two complex square matrices with the same order. If $B \lesssim A$, then $R_B \subseteq \overline{R_A}$, where $\overline{R_A}$ is the topological closure of R_A.*

Proof. Let $z \in R_B$. Then there exists a matrix B_1 in $Q_R(B)$ such that $\det B_1 = z$. Let $A(\varepsilon) = \varepsilon \cdot A + B_1$, it is easy to see that $A(\varepsilon) \in Q_R(A)$ for all $\varepsilon > 0$.

Since $\lim_{\varepsilon \to 0^+} A(\varepsilon) = B_1$, $\lim_{\varepsilon \to 0^+} \det A(\varepsilon) = \det B_1 = z$, which means $z \in \overline{R_A}$. So $R_B \subseteq \overline{R_A}$ holds. $\square$

Corollary 3.1. *Let A be a square matrix. Then $T(A) \subseteq \overline{R_A}$.*

Let C be an arc-weighted directed cycle or a loop. Then it is called an upper cycle if $\mathrm{Im}(\mathrm{ray}(C)) > 0$, a lower cycle if $\mathrm{Im}(\mathrm{ray}(C)) < 0$, a positive cycle if $\mathrm{ray}(C) = 1$, and a negative cycle if $\mathrm{ray}(C) = -1$. Sometimes these are abbreviated as u-cycle, l-cycle, p-cycle and n-cycle.

Next we give some necessary conditions for a matrix A with $R_A = F \cup \{0\} \cup (-F)$, where $\mathrm{ang}(F) < \pi$.

Lemma 3.2. *Let A be a square matrix with order n and all the diagonal elements -1, suppose that $R_A = S_{(0,\theta)} \cup \{0\} \cup S_{(\pi,\theta+\pi)}$, where θ is a fixed real number satisfying $0 < \theta < \pi$. Then:*

(1) *There is no u-cycle in the arc-weighted digraph W(A).*
(2) *There exist some l-cycles in $W(A)$.*
(3) *For any set of pairwise disjoint l-cycles $C_1, \ldots, C_p$ in $W(A)$, write $\mathrm{ray}(C_j) = -e^{i\theta_j} (0 < \theta_j < \pi, j = 1, \ldots, p)$. Then $\sum_{j=1}^{p} \theta_j \leqslant \theta$.*
(4) *There exist some p-cycles in $W(A)$.*
(5) *Any p-cycle is vertex disjoint with any l-cycle.*

Proof

(1) Suppose to the contrary that there exists an u-cycle. Write its ray by $e^{i\theta'}$, where $0 < \theta' < \pi$. Let C be the arc-weighted spanning sub-digraph consisting of this cycle and all the loops, and $A(C)$ be its adjacency matrix, then $A(C) \lesssim A$. Since $(-1)^n$ and $(-1)^n \cdot e^{i(\theta'-\pi)}$ are the only two nonzero terms in the determinantal expansion of the matrix $\mathrm{ray}(A(C))$, we have $R_{A(C)} = (-1)^n \cdot S_{(\theta'-\pi,\, 0)}$. On the other hand, since $0 < \theta < \pi$ and $\overline{R_A} = S_{[0,\theta]} \cup \{0\} \cup S_{[\pi,\theta+\pi]}$, there exists a sufficiently small positive number ε such that $S_{(-\varepsilon,0)} \cap \overline{R_A} = \phi$ and $S_{(-\varepsilon,0)} \subseteq S_{(\theta'-\pi,\, 0)}$, which yields $R_{A(C)} \nsubseteq \overline{R_A}$, contradicting Lemma 3.1.
(2) Since $R_A \backslash \mathbb{R} \neq \phi$, we know that $T(A) \backslash \{\pm 1\} \neq \phi$, there must exist at least one cycle whose ray is not real. By (1) we have the nonexistence of u-cycles, so there must exist an l-cycle.
(3) Denote the arc-weighted digraph consisting of $C_1, \ldots, C_p$ together with all the loops by D, and write $\varphi = \sum_{j=1}^{p} \theta_j$. The adjacency matrix $A(D)$ is ray majorized by A, since D is a sub-digraph of $W(A)$. Since $C_1, \ldots, C_p$ are pairwisely vertex disjoint, we can see that

$$A(D) = (A(C_1) - I_{n_1}) \bigoplus \ldots \bigoplus \left(A(C_p) - I_{n_p}\right) \bigoplus (-I_r),$$

where n_j is the length of the cycle C_j and $n_1 + \cdots + n_p + r = n$. It is not difficult to verify that $R_{A(D)} = (-1)^n \cdot S_{(0,\, \varphi)}$, so together with Lemma 3.1 that $R_{A(D)} \subseteq \overline{R_A}$, we can get $\varphi \leqslant \theta$, implying $\sum_{j=1}^{p} \theta_j \leqslant \theta$.
(4) Since $b_U(R_A) = \{\pm 1,\ \pm e^{i\theta}\} \subseteq T(A)$ (by Theorem 1.1), there exists a permutation cycle set S of A such that the ray of the corresponding nonzero term in the determinantal expansion of A is $(-1)^{n+1} e^{i\theta}$. Suppose the set of all the l-cycles in S is $\mathcal{C}_{-i} = \{C_1, \ldots, C_p\}$, and write $\mathrm{ray}(C_j) = -e^{i\theta_j} (0 < \theta_j < \pi, j = 1, \ldots, p)$, $\varphi = \theta_1 + \cdots + \theta_p$. Similarly, use $\mathcal{C}_1$ to denote the set of all p-cycles in S, and $\mathcal{C}_{-1}$ the set of all n-cycles in S. Then we have

$$\begin{aligned}(-1)^{n+1} e^{i\theta} &= (-1)^n \prod_{C \in S} (-\mathrm{ray}(C)) \\ &= (-1)^n \prod_{C \in \mathcal{C}_{-i}} (-\mathrm{ray}(C)) \prod_{C \in \mathcal{C}_1} (-1) \times 1 \prod_{C \in \mathcal{C}_{-1}} (-1) \times (-1) \\ &= (-1)^n \prod_{j=1}^{p} e^{i\theta_j} \prod_{C \in \mathcal{C}_1} (-1) \\ &= (-1)^n \cdot e^{i\varphi} \prod_{C \in \mathcal{C}_1} (-1).\end{aligned}$$

By (3) we know that $0 < \varphi \leqslant \theta$, by comparing the two sides of the above equation, we see that $\prod_{C \in \mathcal{C}_1}(-1)$ cannot be 1, so $\mathcal{C}_1 \neq \phi$, which shows the existence of p-cycles in S, thus in $W(A)$.

(5) Suppose to the contrary that (5) is not true. Then there exist a p-cycle and an l-cycle containing some common vertices. Choose a pair of such cycles (say, a p-cycle C_1 and an l-cycle C_2) such that the number $|E(C_1) \cup E(C_2)|$ is minimal. Use G to denote the spanning sub-digraph induced by the union of the arc sets $E(C_1), E(C_2)$ and all the loops. Now we claim that all the cycles except C_1 and C_2 in this sub-digraph G are n-cycles.

Suppose G contains some cycle C_3 other than C_1 and C_2, which is not an n-cycle. Then C_3 is either a p-cycle or an l-cycle by (1). Without loss of generality, we may assume C_3 is a p-cycle. Since C_3 is in the sub-digraph G induced by the arc set $E(C_1) \cup E(C_2)$ and all the loops, $C_3 \neq C_1$, $C_3 \neq C_2$, then C_3 contains some arc in C_1 and some arc not in C_1. Thus C_3 must contain a pair of incident arcs $e_1 \in E(C_1)$ and $e_2 \notin E(C_1)$. Without loss of generality, let the common vertex v of e_1 and e_2 be the terminal vertex of e_1 and the initial vertex of e_2. In C_1 denote the arc with v as its initial vertex by e_3, then $e_3 \notin C_2$ and $e_3 \notin C_3$ since $e_2 \in C_2$ and $e_2 \in C_3$. Then C_3 and C_2 is also a pair of p-cycle and l-cycle containing a common arc e_2 (thus containing some common vertices) such that $|E(C_2) \cup E(C_3)| < |E(C_1) \cup E(C_2)|$ since $e_3 \notin E(C_2) \cup E(C_3)$ but $e_3 \in E(D)$, contradicting the choice of C_1 and C_2. So the above claim is true.

Write $\mathrm{ray}(C_2) = -e^{i\theta_2},\ 0 < \theta_2 < \pi$. Since C_1 is a p-cycle, C_2 is an l-cycle, C_1 and C_2 have vertices in common, and all the other cycles in G are n-cycles, then G contains exactly the following three types of permutation cycle set S:

(a) All the cycles in S are n-cycles (loops).
(b) S consists of C_1 and some n-cycles (loops).
(c) S consists of C_2 and some n-cycles (loops).

From this and formula (2.1), we know that $T(A(G))$ contains exactly three different elements, namely $T(A(G)) = \{\pm 1,\ (-1)^n \cdot e^{i\theta_2}\}$, which means $R_{A(E)} = (-1)^n \cdot S_{(0,\,\pi)}$. But by the assumption of this lemma, we have $R_A = S_{(0,\,\theta)} \cup \{0\} \cup S_{(\pi,\,\theta+\pi)} (0 < \theta < \pi)$ and $A(G) \lesssim A$, contradicting Lemma 3.1. $\square$

We are now ready to prove our main result.

Theorem 3.1. *Let A be a complex square matrix of order n. If $R_A = F \cup \{0\} \cup (-F)$, where F is an open sector with* $\mathrm{ang}(F) < \pi$, *then A is partly-decomposable.*

Proof. We only give the proof for the case when n is even (the proof of the other case is similar).

Let $F = S_{(\alpha,\,\alpha+\theta)}$. Then $0 < \theta < \pi$. By Theorem 1.1, we know that $e^{i\alpha} \in b_U(R_A) \subseteq T(A)$. After suitable row permutations, we may assume that the product of the n diagonal elements of A is $e^{i\alpha}$ (or $-e^{i\alpha}$). Multiplying each row of A by suitable nonzero complex numbers, we may further assume that all the diagonal elements of A are -1, also we still have $1 = (-1)^n \in b_U(R_A)$. Thus we may assume that $\alpha = 0$, and so $F = S_{(0,\,\theta)}$.

Now we prove the contrapositive. Suppose that A is fully-indecomposable, then the arc-weighted associating digraph $W(A)$ is strongly connected.

According to Lemma 3.2, $W(A)$ contains a p-cycle and an l-cycle which are vertex disjoint. So by Lemma 2.2, there is a cycle chain in $W(A)$ with a p-cycle and an l-cycle as its two end cycles. Choose a shortest such cycle chain, say $C_1, \ldots, C_q$ $(q \geq 3)$, where $\mathrm{ray}(C_1) = 1$ and $\mathrm{ray}(C_q) = -e^{i\theta'}$ $(0 < \theta' \leqslant \theta)$. Since it is the shortest one and there is no u-cycle in $W(A)$ (by Lemma 3.2), we can deduce that $\mathrm{ray}(C_j) = -1$ for all $j = 2, \ldots, q-1$.

Denote W_1 to be the arc-weighted spanning sub-digraph of $W(A)$ determined by the arc sets of $C_1, \ldots, C_q$ together with all the loops, and write $B = A(W_1)$. Then we have $B \lesssim A$. Now we want to show that $R_B \nsubseteq \overline{R_A}$ (which would contradict Lemma 3.1).

First we show that B is ray nonsingular.

Let $B' \in Q_R(B)$, multiply each row of B' by suitable positive numbers, we may assume that all the diagonal entries of B' are -1. Let w_j $(j = 1, 2, \ldots, q)$ be the weight of the cycle C_j in the arc-weighted digraph $W(B')$.

We still use the notation of Lemma 2.3. Let N_0 be the arc-weighted spanning sub-digraph of $W(B')$ consisting of n loops with weight -1, $N_j = N_{j-1} \cup C_j, j = 1, 2, \ldots, q$. In particular, $N_q = W(B')$.

Since the rays of the cycles $C_1, \ldots, C_{q-1}$ and the loops are all real, and $C_1, \ldots, C_j$ are all the non-loop cycles of N_j, we know that all terms in the determinantal expansion of $A(N_j)$ are real, so $\det N_j$ is real $(0 \leqslant j \leqslant q-1)$.

Using Lemma 2.3 together with the hypothesis that n is even, we can get the following formulas:

$$\det N_0 = 1, \quad \det N_1 = 1 - |w_1|,$$

$$\det N_j = |w_j| \cdot \det N_{j-2} + \det N_{j-1} \quad (j = 2, \ldots, q-1), \tag{2.3}$$

$$\det B' = e^{i\theta'} \cdot |w_q| \cdot \det A(N_{q-2}) + \det A(N_{q-1}). \tag{2.4}$$

Now we use (2.4) to prove $\det B' \neq 0$, 1 and the complex number $e^{i\theta'}$ are linearly independent over the real field, so we only need to prove that the two real number $\det N_{q-2}$ and $\det N_{q-1}$ cannot both be zero, which is equivalent to say that $\begin{pmatrix} \det N_{q-1} \\ \det N_{q-2} \end{pmatrix}$ is not a zero vector.

Rewrite the recurrence formula (2.3) in the following matrix form:

$$\begin{pmatrix} \det N_j \\ \det N_{j-1} \end{pmatrix} = \begin{pmatrix} 1 & |w_j| \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \det N_{j-1} \\ \det N_{j-2} \end{pmatrix} \quad (j = 2, \ldots, q-1).$$

So we have

$$\begin{pmatrix} \det N_{q-1} \\ \det N_{q-2} \end{pmatrix} = \left(\prod_{j=2}^{q-1} \begin{pmatrix} 1 & |w_j| \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 - |w_1| \\ 1 \end{pmatrix}.$$

Since $w_j \neq 0$, the square matrix $\begin{pmatrix} 1 & |w_j| \\ 1 & 0 \end{pmatrix}$ is nonsingular $(j = 2, \ldots, q-1)$. So $\prod_{j=2}^{q-1} \begin{pmatrix} 1 & |w_j| \\ 1 & 0 \end{pmatrix}$ is nonsingular. Also $\begin{pmatrix} 1 - |w_1| \\ 1 \end{pmatrix}$ is not a zero vector, so $\begin{pmatrix} \det N_{q-1} \\ \det N_{q-2} \end{pmatrix}$ is not a zero vector. It follows from (2.3) that $\det B' \neq 0$.

Since $\det B' \neq 0$ for all $B' \in Q_R(B)$, we know that B is ray nonsingular.

Now in the arc-weighted spanning sub-digraph W_1 of $W(A)$ defined above (here $B = A(W_1)$), we have $\text{ray}(C_1) = 1$, $\text{ray}(C_q) = -e^{i\theta'}$, C_1 and C_q are vertex disjoint and the ray of all other cycles $C_2, \ldots, C_{q-1}$ and the n loops are -1. By the formula (2.1), it is easy to verify that $T(B) = \{\pm 1, \pm e^{i\theta'}\}$, which implies that R_B is not contained in a line.

On the other hand, since B has already been proved to be ray nonsingular, so $R_B = R_B \backslash \{0\}$ is open and connected, it follows that R_B is an open sector, say, $R_B = S_{(\beta,\gamma)}$. By Corollary 3.1, $T(B) \subseteq \overline{R_B} = S_{[\beta,\gamma]} \cup \{0\}$, which means $\text{ang}(R_B) > \pi$. It follows that $R_B \nsubseteq \overline{R_A}$, since $R_A = S_{(0,\theta)} \cup \{0\} \cup S_{(\pi,\pi+\theta)}$, contradicting Lemma 3.1. Thus A is partly-decomposable, completing the proof of the theorem. □

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